Section 4.1

Extreme Values

It is now our goal to use derivatives to analyze the behavior of functions. The first points of study are extrema.

Definition 4.1.1. Let f be defined on a set D containing c. If $f(c) \geq f(x)$ for all x in D, then $f(c)$ is an *absolute maximum* of f on D. If $f(c) \leq f(x)$ for all x in D, then $f(c)$ is an *absolute minimum*. An absolute extreme value is either an absolute maximum or an absolute minimum.

Absolute extrema depend on both the function f and the set D. Consider $f(x) = 4 - x^2$ on $D_1 = (-\infty, \infty), D_2 = [0, 2], D_3 = (0, 2]$, and $D_4 = (0, 2)$.

Theorem 4.1.2 (Extreme Value Theorem)**.** If f is continuous on $[a, b]$ then there are numbers c, d between a and b (including a and b) such that $f(c)$ is an absolute maximum and $f(d)$ is an absolute minimum.

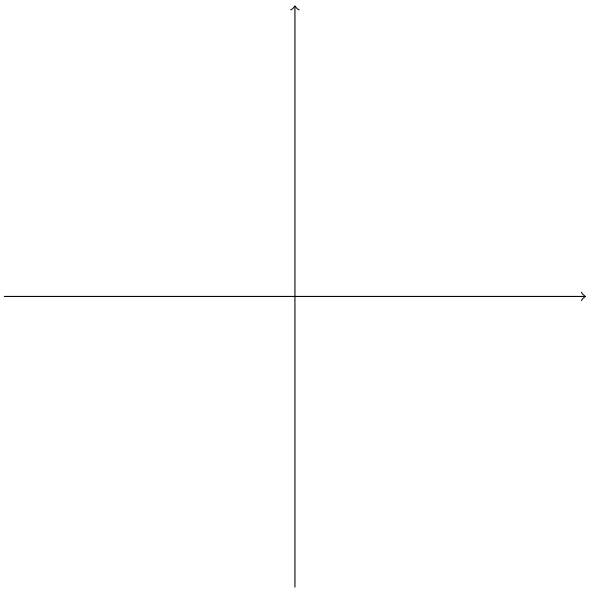
Let's look at extreme values on graphs of functions.



Local Extrema

Definition 4.1.3. Suppose f is defined for all values near c. If there is some open interval I containing c such that $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a *local maximum* of f. If there is some open interval J containing c such that $f(c) \leq f(x)$ for all x in J, then $f(c)$ is a *local minimum* of f.

We have already located some absolute extrema via graphs, continue now with local extrema:



Theorem 4.1.4 (Fermat's Theorem)**.** If f has a local extremum at c and $f'(c)$ exists, then $f'(c) = 0$.

Local extrema can also occur at points where f is not differentiable as was illustrated above. This makes two types of points important, both described by the derivative of f.

Definition 4.1.5. A point c interior to the domain of f at which $f'(c) = 0$ or $f'(c)$ does not exist is called a *critical point* of f.

Local extrema occur at critical points, but not all critical points are local extrema.

Example 4.1.6.



This gives us a procedure to find the absolute extrema for continuous functions (which must exist due to EVT).

Procedure 4.1.7 (Locating Absolute Extreme Values)**.** Suppose f is continuous on $[a, b]$.

1. Locate the critical points c in (a, b). These points are candidates for absolute extrema.
2. Evaluate f at each critical point and the endpoints of $[a, b]$.
3. The largest and smallest values from Step 2. are the absolute maximum and absolute minimum, respectively.

Example 4.1.8. Determine the location and value of the absolute extrema of the following:

1. $f(x) = x^2 - 10$ on $[-2, 3]$

2. $f(x) = (x + 1)^{4/3}$ on $[-9, 7]$

3. $f(x) = \cos^2(x)$ on $[0, \pi]$

4. $f(x) = \dfrac{x}{(x^2+3)^2}$ on $[-2, 2]$